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A MODEL FOKKER-PLANCK EQUATION FOR A PLASMA AND ITS SOLUTION

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ABSTRACT

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We discuss a model Fokker-Planck equation, (14), for a single species of particles in a plasma. This equation has several properties in common with the real equation, and ascribes an approximately correct behavior to most of the particles, though incorrect for the high energy tail in the thermal distribution. It is shown that the equation can be solved completely, for the small perturbations of a uniform plasma by electric fields harmonic in space and time, in an external magnetic field. Applications to ionospheric radar scattering are briefly discussed; it is shown that in certain circumstances ion-ion collisions can have a profound effect on the scattering even though the collision-frequency is much smaller than the ion gyrofrequency, and this appears to agree with observation. Author

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I. INTRODUCTION

Some problems in plasma physics require that one takes account of collisions but not to the extent that the dynamics is collision dominated. In such a case, neither the "Vlasov" equations nor the hydrodynamic equations (even treating each species as a distinct fluid) are adequate; the former would omit the collisions, so including only the collective interactions, while the latter would omit subtleties, such as Landau damping, which arise only when distributions in velocity space are considered. So we have to write down, for each species, an equation such as

$$\frac{\partial f}{\partial t} + v_j \frac{\partial f}{\partial x_j} + a_j \frac{\partial f}{\partial v_j} = \left(\frac{\partial f}{\partial t}\right)_c \tag{1}$$

where $f(\underline{x}, \underline{v}, t)$ is the Boltzmann function in the usual notation, $\underline{a}$ is the macroscopic contribution to the acceleration is a particle at $(\underline{x}, \underline{v}, t)$ and the term on the right represents collisions. When there are large-angle binary collisions, $(\partial f/\partial t)_c$ is the usual collision integral due to Boltzmann, so is known but leads to formidable analysis in the type of problem we have in mind. When the collisions are not predominantly binary, the actual formula for $(\partial f/\partial t)_c$ in the general case is not known, though in the case of a spatially uniform plasma considerable progress has been made by numerous authors. The expressions generally given are of the Fokker-Planck type

$$\left(\frac{\partial f}{\partial t}\right)_c = \frac{\partial}{\partial v_i}\left\{-A_i f + \frac{1}{2}\frac{\partial}{\partial v_j}(B_{ij} f)\right\} \tag{2}$$

Here A_i and B_{ij} are the "friction" and "diffusion" coefficients respectively; besides being functions of the $\underline{v}$ at which $(\partial f/\partial t)_c$ is calculated, they are also complicated functionals of f itself. When the plasma is not uniform, as for example in plasma oscillations, the calculations of A_i and B_{ij} becomes extremely difficult.

A procedure, due apparently to Bhatnagar, Gross and Krook (1954) for making progress in problems governed by equation (1), is to replace the right hand side by a simple expression qualitatively similar to the correct one, and constructed in such a way that certain conservation laws are not violated. It must also ensure that the distribution function tends to the one appropriate for thermal equilibrium if the gas is isolated. The equation these authors used for this purpose was

$$\left(\frac{\partial f}{\partial t}\right)_c = -\nu(f-f_{max}) \tag{3}$$

where f_{max} is a suitable maxwellian distribution and ν is a constant which may be called the "collision frequency". This has come to be known as the BGK model. It is suitable for representing the effects of large angle binary collisions, so its applicability to plasma dynamics is restricted to the case when it is the collisions with neutral particles which are of interest. Bhatnagar, Gross and Krook (1954) and Gross and Krook (1956) treated the problem of small-amplitude waves propagating in a gas. The corresponding problem with the addition of a uniform magnetic field was worked out by Dougherty (1963) and by Lewis and Keller (1962). The results have found applications to problems concerning the ionosphere (Dougherty and Farley, 1963; Farley, 1963a, 1963b) with considerable agreement with experiment.

When the collisions to be described are those between charged particles it is the "grazing" collisions which make the major contribution. These occur when the impact parameter lies between the interparticle spacing $(N^{-1/3})$ and the Debye length; they are "small-angle" and are not binary. To handle these, we naturally ask whether an equation of form (2) but suitably simplified, could be used as a model in a way analogous to the work just mentioned. The incentive to do this again springs from ionospheric matters. In the theory of incoherent scatter the spectrum of density fluctuations in thermal equilibrium is required, and the various methods which have been used all lead eventually to the solution of the Boltzmann equation. One prediction of the collision-free theory is that for a fixed wave number $\underline{k}$ nearly orthogonal to an imposed field $\underline{B}$, the frequency spectrum has resonant peaks at multiples of the gyrofrequency; here it is the ion gyro-frequency which is relevant because of electrostatic control by the ions (Fejer, 1961; Salpeter, 1961; Hagfors, 1961; Farley, Dougherty and Barron, 1961). Despite careful search, these resonant peaks have not so far been observed (Bowles, private communication). In the lower ionosphere the ion-neutral collisions would be the most important, and using the BGK model, Doughterty and Farley (1963) have shown how the resonances could be smoothed out by the collisions. But in the F region another explanation is required, and Farley (private communication) has given a physical description of a mechanism based on ion-ion collisions which it is estimated may be responsible for the loss of the gyro-resonances. Paradoxically, Colin, Burns and Eshleman (1963) have actually detected a weak gyro-resonance in another type of scattering; it seems to be present in the E region at night but has not so far been detected in any other circumstances.

Our object here is to formulate an equation of the type (2) and show how a formal solution can be obtained for small perturbations harmonic in space and time, with the inclusion of a uniform external magnetic field. This solution is suitable for numerical work for use in the applications just mentioned. We shall restrict attention to a single species of charged particles, and in dealing with ions this is adequate as ion-electron collisions will have a negligible effect on the ions. Corresponding work for several species is of course similar in principle but even more cumbersome.

II. FORMULATION OF THE EQUATION

We wish to choose coefficients A_i, B_{ij} for (2) satisfying the following conditions.

(a) The number density, momentum and energy at each point in physical space shall be conserved. This requires

$$\int \left(\frac{\partial f}{\partial t}\right)_c d^3v = 0 \qquad (4)$$

$$\int \left(\frac{\partial f}{\partial t}\right)_c v_k \, d^3v = 0 \qquad (5)$$

$$\int \left(\frac{\partial f}{\partial t}\right)_c v^2 \, d^3v = 0 \qquad (6)$$

where $v^2 = v_k v_k$ in the usual suffix notation.

(b) The only solution of $(\partial f/\partial t)_c = 0$ shall be the maxwellian distribution, where the number density, drift velocity and temperature are arbitrary.

We assume throughout that as $v \to \infty$ in any direction, $f \to 0$ faster than any power of v, and that A_i, B_{ij} are rational functions of v; then in any integrations over velocity space contributions at infinity are zero. Eq. (4) is therefore automatically satisfied.

We now propose to take

$$A_i = -\nu(v_i - u_i) \qquad (7)$$

$$B_{ij} = \frac{2\nu KT}{m}\delta_{ij} \qquad (8)$$

where ν is an inverse time, independent of $\underline{v}$, m is the mass, and $\underline{u}$, T are the local drift velocity and temperature respectively:

$$Nu_i = \int v_i f d^3v \qquad (9)$$

$$3NKT = \int m(\underline{v}-\underline{u})^2 f d^3v \qquad (10)$$

N being the local number density

$$N = \int f d^3v \qquad (11)$$

and K is Boltzmann's constant. That (5) and (6) are satisfied is easily shown. Further we note that a maxwellian distribution with drift satisfies

$$\frac{\partial f}{\partial v_j} = -\frac{m}{KT}(v_j - u_j)f \tag{12}$$

from which it follows at once that condition (b) is satisfied.

This choice of A_i, B_{ij} appears to be the simplest one available. It makes the friction coefficient proportional to the velocity relative to the mean velocity, and makes the diffusion coefficient isotropic and independent of velocity. This is indeed quite a good approximation for most of the particles, but it is incorrect for the high-energy "tail" in the distribution. For fact particles the friction ought to decrease with velocity.

The choice of an isotropic B_{ij} imposes a restriction on our model. The perturbations in velocity which B_{ij} represents arise from all the particles within the Debye sphere for the particle under observation, and should really be calculated by considering the stochastic electric field set up within the sphere (Thompson and Hubbard, 1960). Our assumption is that the spectrum of electric field fluctuations is isotropic <u>at this length scale</u>. The condition for this is that the Debye length be much smaller than the Lamor radius for a thermal particle; this is usually the case.

The coefficients A_i, B_{ij} which we have defined are functionals of f inasmuch as $\underline{u}$, T depend on f through Eqs. (9) and (10). This is also a property of the real kinetic equation, and it makes the problem of solving it a non-linear one. Further, we have to consider how ν is to be calculated; though independent of $\underline{v}$ in this model, it too may be a functional of f, and one would certainly expect it to depend on N, for instance. This piece of information is not, however, required in what follows, as we treat only the problem of small perturbations about a uniform state, so that ν becomes constant. We shall rather loosely call ν the "collision frequency." In reality, a consideration of the way (2) arises shows that $1/\nu$ is the time in which a thermal particle can expect to suffer a substantial change of velocity. Such a change is actually achieved by a succession of many small increments, which occur at a much greater frequency than ν. For numerical purposes in connection with the positive ions in the ionosphere, we can identify $1/\nu$ with t_c as defined by Spitzer (1956), Eq. (5.26) namely

$$\nu^{-1} = t_c = \frac{11.4 \cdot A^{1/2}\, T^{3/2}}{N \log \Lambda} \text{ sec} \tag{13}$$

where A is the atomic weight of the ions, N is the density in cm^{-3}, and $\Lambda = \frac{3}{2e^3}(K^3T^3/\pi N)^{1/2}$. Actually $\log \Lambda$ is very slowly varying, and is about 13 for typical ionospheric conditions.

Our kinetic equation is, finally,

$$\frac{\partial f}{\partial t} + v_j \frac{\partial f}{\partial x_j} + a_j \frac{\partial f}{\partial v_j} = \nu \frac{\partial}{\partial v_j}\left\{(v_j - u_j)f + \frac{KT}{m}\frac{\partial f}{\partial v_j}\right\} \tag{14}$$

III. SMALL PERTURBATIONS

Following the usual course we write $f = f_o + f_1$ where f_o is a maxwellian distribution with $N = N_o$, $T = T_o$ and zero drift velocity, while f_1 is a small perturbation. The external acceleration in the unperturbed state is supposed to be just that due to a uniform magnetic field, $\underline{B}$. The small additional acceleration in the perturbed state will now be called $\underline{a}$. We write $N = N_o + N_1$, $T = T_o + T_1$ and note that $\underline{u}$ is a perturbation quantity. As mentioned above f_o makes the right hand side of (14) vanish identically for any value of ν, so we may let ν have its unperturbed value during the perturbations. So to the first order, the right hand side of (14) is

$$\left(\frac{\partial f}{\partial t}\right)_c \approx \nu \frac{\partial}{\partial v_j}\left\{v_j f_1 - u_j f_o + \frac{K}{m}\left(T_o \frac{\partial f_1}{\partial v_j} + T_1 \frac{\partial f_o}{\partial v_j}\right)\right\}$$

while the acceleration term on the left becomes

$$\frac{e}{mc}(\underline{v} \times \underline{B})_j \frac{\partial f_1}{\partial v_j} + a_j \frac{\partial f_o}{\partial v_j}$$

Writing in explicitly the derivatives of f_o (Eq. (12) with zero drift) and collecting the terms in f_o and f_1 on the two sides of the equation, (14) becomes

$$\frac{\partial f_1}{\partial t} + v_j \frac{\partial f_1}{\partial x_j} + \frac{e}{mc} (\underline{v} \times \underline{B})_j \frac{\partial f_1}{\partial v_j} - \nu \frac{\partial}{\partial v_j} \left\{ v_j f_1 + \frac{KT_o}{m} \frac{\partial f_1}{\partial v_j} \right\}$$

$$= \frac{m}{KT_o} a_j \, v_j f_o + \nu \left\{ \frac{m u_j v_j u}{KT_o} + \frac{T_1}{T_o} \left(\frac{mv^2}{KT_o} - 3 \right) \right\} f_o \qquad (15)$$

Here we have a linear equation for f_1 but we have to remember that $\underline{u}$ and T_1 are integrals over f_1; in fact on taking perturbations of (9) - (11) we have

$$N_1 = \int f_1 d^3 v \qquad (16)$$

$$u_i = \frac{1}{N_o} \int f_1 v_i d^3 v \qquad (17)$$

$$\frac{T_1}{T_o} = \frac{1}{N_o} \left[\frac{m}{3KT_o} \int v^2 f_1 d^3 v - N_1 \right] \qquad (18)$$

IV. FORMAL SOLUTION

The procedure for dealing with these integral terms is the same as in the BGK model. For a disturbance harmonic in space and time we first solve (15) for f_1 supposing the right hand side given; this is solely a problem in velocity space and $\underline{u}$, T_1 are merely coefficients. When this solution is substituted into (16) - (18) we have a set of equations for f_1, N_1, $\underline{u}$ and T_1 and then the solution for any of these quantities is only a matter of algebra. Writing the left hand side of (15) as $\mathcal{D} f_1$ where $\mathcal{D}$ is a linear differential operator, suppose we can solve any equation $\mathcal{D} f_1 = h$ where h is harmonic in space and time, so that symbolically $f_1 = \mathcal{D}^{-1}h$. The solution of (15) for f_1 is

$$f_1 = \frac{m}{KT_o}(a_j + \nu u_j)\,\mathcal{D}^{-1}(v_j f_o) + \frac{\nu T_1}{T_o}\left[\frac{m}{KT_o}\,\mathcal{D}^{-1}(v^2 f_o) - 3 \quad {}^{-1}f_o\right] \tag{19}$$

Let us define a set of quantities

$$_{i},\ \ldots H_{j}\ldots = \int v_i \ldots \mathcal{D}^{-1}(v_j \ldots f_o)\,d^3v \tag{20}$$

where we shall need at most 2 suffixes before or after H. Each suffix (if any) labels the component of $\underline{v}$ to be inserted in the appropriate place in the integral. In this notation Eqs. (16) - (18) are

$$N_1 = \frac{m}{KT_o}(a_j + \nu u_j)H_j + \frac{\nu T_1}{T_o}\left(\frac{m}{KT_o}H_{jj} - 3H\right) \tag{21}$$

$$N_o u_i = \frac{m}{KT_o}(u_j + \nu u_j)\,{}_iH_j + \frac{\nu T_1}{T_o}\left(\frac{m}{KT_o}\quad {}_iH_{jj} - 3\,{}_iH\right) \tag{22}$$

$$N_1 + N_o\frac{T_1}{T_o} = \frac{m}{3KT_o}\left\{\frac{m}{KT_o}(a_j + \nu u_j)\,{}_{ii}H_j + \frac{\nu T_1}{T_o}\left(\frac{m}{KT_o}\,{}_{ii}H_{jj} - 3\,{}_{ii}H\right)\right\} \tag{23}$$

Here we have explicit equations for N_1, $\underline{u}$ and T_1, and insertion of $\underline{u}$ and T_1 into (19) would make the latter an explicit equation for f_1. However, for many purposes the details of f_1 are not in fact of interest; in particular in plasma physics we need only N_1 and $\underline{u}$ to find the contribution which a species makes to the charge and current densities and so combine the dynamics with maxwell's equations. As N_1 is given simply in terms of div $\underline{u}$, we solve (21) - (23) for $\underline{u}$ alone. The result is

$$N_o u_i = M_{ij}(a_j + \nu u_j) \tag{24}$$

where

$$M_{ij} = \frac{m}{KT_o}\left\{{}_iH_j + \frac{3\nu\left(\frac{m}{3KT_o}\,{}_iH_{pp} - {}_iH\right)\left(\frac{m}{3KT_o}\,{}_{qq}H_j - H_j\right)}{N_o - 3\nu\left\{\left(\frac{m}{3KT_o}\right)^2 {}_{pp}H_{qq} - \frac{m}{3KT_o}\left({}_{pp}H + H_{qq}\right) + H\right\}}\right\} \tag{25}$$

and solving (24) explicitly, in matrix notation

$$N_o\underline{u} = \left(I - \frac{\nu}{N_o}\underline{M}\right)^{-1} M\underline{a} \tag{26}$$

where I is the unit 3×3 matrix. It is to be noted from (25) that only certain contractions of all the possible H-functions (with anything from 0 to 4 suffixes) are required. But first we must develop a general technique for constructing these functions.

V. PROCEDURE FOR SOLVING BOLTZMANN'S EQUATION

Let us take as our standard form of Eq. (15)

$$\frac{\partial f}{\partial t} + v_j\frac{\partial f}{\partial x_j} + \frac{e}{mc}(\underline{v}\times\underline{B})_j\frac{\partial f}{\partial v_j} - \nu\frac{\partial}{\partial v_j}(v_j f) - \eta\frac{\partial^2 f}{\partial v_j \partial v_j} = h(\underline{x},\underline{v},t) \tag{27}$$

where $\eta = \nu KT_o/m$, h is a given function and we omit the suffix from f_1 in what follows.

Two general routes appear here. In the first we impose at once the harmonic variations in $\underline{x}$ and t and attempt to solve the differential equation in velocity space with appropriate choices of h, finally evaluating the integrals (20). This is the procedure used by Landau (1946), Berstein (1958) and many others, in the case of no collisions. Although it seems inviting, it is a good deal more difficult in our problem, as (27) is now second order in $\underline{v}$, with its coefficients dependent on $\underline{v}$. Actually these coefficients are linear in $\underline{v}$ and a generalization of Laplace's contour integral method (Burkill, 1956) may be used, the calculation looking superficially like a Fourier transform with respect to $\underline{v}$. Eq. (27) has been considered in this way for the case of no magnetic field by Lenard and Bernstein (1958, unpublished).

The second route to be followed here is to solve first (27) in the case when h is a unit source at a point $(\underline{x}_o, \underline{v}_o)$, at time t_o, i.e., when

$$h(\underline{x},\underline{v},t) = \delta^3(\underline{x}-\underline{x}_o)\ \delta^3(\underline{v}-\underline{v}_o)\ \delta(t-t_o) \tag{28}$$

or equivalently to set $h = 0$ for $t > t_o$ and to solve with the initial value $f = \delta^3(\underline{x}-\underline{x}_o)\ \delta^3(\underline{v}-\underline{v}_o)$. We call this solution $G(\underline{x},\underline{v},t,\underline{x}_o,\underline{v}_o,t_o)$, the Green's function. The solution for a general h is then the convolution of G with h, a 7-fold integral. The quantities $_i\ldots H_j\ldots$ are therefore 10-fold integrals. In the absence of collisions G is simply a 6-fold δ-function transferring the source from $(\underline{x}_o\underline{v}_o)$ to a point $(\underline{x},\underline{v})$, these two points being on the same "unperturbed orbit" and at times t_o, t respectively. This follows from the fact that with $\nu = \eta = 0$ (27) is only first order, so f is constant along the characteristic curves

$$dt = \frac{dx_j}{v_j} = \frac{dv_j}{\frac{e}{mc}\,(\underline{v}\times\underline{B})_j} \tag{29}$$

which are just the unperturbed orbits. Thus most of the 10 integrations exist only in a trivial sense and the formulae are easily reduced to a single integral, whose physical significance is a summing of perturbations over the past histories of particles.

When collisions are included the Green's function is not a δ-function. A perturbation is not simply convected along an unperturbed orbit, but spreads out owing to the term $-\eta\ \partial^2 f/\partial v_j \partial v_j$. As one might guess from the origin of this diffusion term as a stochastic process in velocity space, G becomes a Gaussian distribution in $\underline{x}$ and $\underline{v}$ centered on the instantaneous position on the unperturbed orbit. The coefficients of this Gaussian function are somewhat cumbersome, though elementary functions of t, and the "unperturbed orbit" must now take account of a frictional contribution $-\nu v_j$ to the acceleration. The form of G itself gives some insight into the behavior of the gas, and we shall also see that it is again possible to reduce the 10-fold integral to a single one. Unfortunately quite a number of cases of this integral will arise because of the various types of H needed in (25).

VI. CONSTRUCTION OF THE GREEN'S FUNCTION

In this section the particular form taken by the coefficients in (27) is irrelevant; all that matters is that the coefficients of the second derivatives are constant and those of the first derivatives linear in the independent variables. It will therefore be clearer to consider the general equation of form

$$\frac{\partial f}{\partial t} + a_{ij}\frac{\partial^2 f}{\partial x_i \partial x_j} + b_{ij}x_j\frac{\partial f}{\partial x_i} + cf = 0 \tag{30}$$

where x_i $(i=1,2,\ldots N)$ represents all the independent variables other than t (so including $\underline{x}$ and $\underline{v}$ in our application, with $N=6$). The coefficients a_{ij}, b_{ij} and c are all constants, a_{ij} being symmetric. We take care not to assume that a_{ij}, b_{ij} are non-singular matrices; they are actually singular in our problem. We see that when $h = 0$ in (27), that equation takes the form (30) in this notation; further we have $c = b_{ii}$, which ensures that $\int f d^N x$ is a constant.

The problem is to find $f(\underline{x},t)$ such that $f \to \delta^N(\underline{x}-\underline{y})$ as $t \to 0$ for any given $\underline{y}$, so we have a unit source of f at $\underline{x}=\underline{y}$ and $t=0$.

First we note that provided we can construct all the loci $\underline{Y}(t)$ satisfying

$$\frac{dY_i}{dt} = b_{ij}Y_j \tag{31}$$

the problem is reduced to the case of a source at the origin, i.e., $\underline{y} = 0$. In the plasma problem these loci are just what we called the "unperturbed orbit." For let $g(\underline{x},t) = g(\underline{x}-\underline{Y}(t-t'),t-t')$ be the solution for unit source at $\underline{y}$ at time $t = t'$ if $\underline{y}(t)$ is the orbit fixed by $\underline{Y}(0) = \underline{y}$. This solution obviously has the right initial condition, and noting that

$$\frac{\partial f}{\partial t} = \frac{\partial g}{\partial t} - \frac{\partial g}{\partial x_i} \frac{dY_i}{dt}$$

the left hand side of (30) becomes

$$\frac{\partial g}{\partial t} - \frac{\partial g}{\partial x_i} \frac{dY_i}{dt} + a_{ij} \frac{\partial^2 g}{\partial x_i \partial x_j} + b_{ij} x_j \frac{\partial g}{\partial x_i} + cg$$

where g and its derivatives are evaluated at $(\underline{x}-\underline{Y}(t-t'), t-t')$. We can write this

$$\left[\frac{\partial g}{\partial t} + a_{ij} \frac{\partial^2 g}{\partial x_i \partial x_j} + b_{ij}(x_j - Y_j) \frac{\partial g}{\partial x_i} + cg\right] + \frac{\partial g}{\partial x_i} \left[b_{ij} Y_j - \frac{dY_i}{dt}\right]$$

The first square bracket is zero for $t > t'$ by construction of g. The second is zero by construction of $\underline{Y}$.

To find this standard solution $g(\underline{x}, t)$ we assume the form

$$\log g = \log p - \tfrac{1}{2} q_{ij} x_i x_j \tag{32}$$

where p, q_{ij} are functions of t only and the quadratic form $q_{ij} x_i x_j$ is to be non-negative (the factor $\tfrac{1}{2}$ is merely conventional). On evaluating the various derivatives, substituting them into (30) (g playing the part of f) and dividing through by g one obtains eventually

$$\left\{\frac{1}{p} \frac{dp}{dt} - a_{ij} q_{ij} + c\right\} + \left\{-\tfrac{1}{2} \frac{dq_{k\ell}}{dt} + a_{ij} q_{kj} q_{\ell i} - b_{i\ell} q_{ki}\right\} x_k x_\ell = 0$$

For this to hold identically in $\underline{x}$, t, the first bracket must vanish, and the second bracket symmetrized with respect to k, ℓ must vanish, so we have an equation for p (which we shall not actually need), together with an equation for q which in matrix notation is

$$-\frac{dq}{dt} + 2\underline{q}\ \underline{aq} - \underline{q}\ \underline{b} - \underline{b}'\underline{q} = 0 \tag{33}$$

Here $\underline{b}'$ denotes the transpose of $\underline{b}$; $\underline{a}$ and $\underline{q}$ are of course already symmetric. Writing $\underline{r} = \underline{q}^{-1}$ leads at once to the linear equation

$$\frac{dr}{dt} + 2\underline{a} - \underline{b}\ \underline{r} - \underline{r}\ \underline{b}' = 0 \tag{34}$$

We mentioned in the previous section that for the plasma problem one would expect that g would be a multivariate normal distribution, such as (32); from this point of view the coefficients $\underline{r}$ are more significant than $\underline{q}$, for they are just the variances and correlations of all the variables concerned. Using this idea a slightly shorter derivation of (34) arises if we assume the form (32) and integrate (30) over all $\underline{x}$-space after multiplying by suitable components of $\underline{x}$. However, the derivation outlined above actually verifies that (32) is applicable. We observe also that the initial conditon for $\underline{r}$ is clearly $\underline{r} = 0$ at $t = 0$, making all the variances and correlations zero. Finally in the case when $c = b_{ii}$, so $\int f d^N x$ is constant (and equal to unity for our g) we need not solve the equation for $p(t)$, for we have

$$p(t) = [(2\pi)^N \det(\underline{r})]^{-\frac{1}{2}} \tag{35}$$

The solution of (34) presents no difficulty in any specific case, but the form of the solution depends on the ranks of $\underline{a}$ and $\underline{b}$, as one can see by imagining a change of coordinates to make $\underline{a}$ and $\underline{b}$ diagonal. Rather than enumerate the cases which could arise we deal later with the particular one we need in the application.

To complete the general theory of an equation such as

$$\frac{\partial f}{\partial t} + a_{ij}\frac{\partial^2 f}{\partial x_i \partial x_j} + b_{ij}x_j\frac{\partial f}{\partial x_j} + cf = h(\underline{x},t) \tag{36}$$

we may now write down the solution for any problem in which $f \to 0$ as $t \to \infty$ and $h(\underline{x},t)$ is given. It is the convolution

$$f(\underline{x},t) = \int_{\underline{y}} \int_{t'=-\infty}^{t} h(\underline{y},t')\, G(\underline{x},t,\underline{y},t')\, d^N y dt' \tag{37}$$

where

$$G(\underline{x},t,\underline{y},t') = g[\underline{x}-\underline{Y}(t-t'),t-t'] \tag{38}$$

and

$$g(\underline{x},t) = p(t) \exp\left\{-\tfrac{1}{2}\, q_{ij}x_i x_j\right\} \tag{39}$$

is the standard solution just derived, with $\underline{q} = \underline{r}^{-1}$, $\underline{r}$ given by (34), p by (35) and $Y(t)$ is the solution of (31) with $\underline{Y} = \underline{y}$ at $t = 0$. Writing $\tau = t-t'$, (37) is

$$f(x,t) = \int_{\underline{y}} \int_{\tau=0}^{\infty} h(\underline{y},t-\tau)\, g(\underline{x}-\underline{Y}(\tau),\tau) d^N y d\tau \tag{40}$$

VII. THE PLASMA PROBLEM IN ONE DIMENSION WITHOUT MAGNETIC FIELD

As an example of the above process, let us consider the equation governing a plasma, (27) reduced to one dimension without magnetic field, i.e.,

$$\frac{\partial f}{\partial t} - \eta \frac{\partial^2 f}{\partial v^2} + v \frac{\partial f}{\partial x} - \nu v \frac{\partial f}{\partial v} - \nu f = h(x,v,t) \tag{41}$$

Thus $\underline{x}$ becomes $\binom{x}{v}$ and

$$\underline{a} = \begin{pmatrix} 0 & 0 \\ 0 & -\eta \end{pmatrix} \qquad \underline{b} = \begin{pmatrix} 0 & 1 \\ 0 & -\nu \end{pmatrix} \qquad c = -\nu$$

Write $\underline{r} = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}$; then equation (34) is

$$\frac{d}{dt}\begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} - 2\gamma \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 2\beta & \gamma-\nu\beta \\ \gamma-\nu\beta & -2\nu\gamma \end{pmatrix} = 0$$

i.e.,

$$\begin{cases} \alpha - 2\beta = 0 \\ \beta + \nu\beta - \gamma = 0 \\ \gamma + 2\nu\gamma - 2\eta = 0 \end{cases}$$

These are readily solved for γ, β, α in turn, each to be zero at $t = 0$, giving

$$\alpha = \frac{2\eta}{\nu^3}\left(\nu t + 2e^{-\nu t} - \frac{1}{2}e^{-2\nu t} - \frac{3}{2}\right)$$

$$\beta = \frac{\eta}{\nu^2}\left(1 - 2e^{-\nu t} + e^{-2\nu t}\right) \qquad (42)$$

$$\gamma = \frac{\eta}{\nu}\left(1 - e^{-2\nu t}\right)$$

and one would have

$$p(t) = \frac{1}{2\pi(\alpha\gamma-\beta^2)^{\frac{1}{2}}}$$

We note that for $t \ll \nu^{-1}$, $\alpha \simeq 2\eta t^3/3$, $\beta \simeq \eta t^2$, $\gamma \simeq 2\eta t$.

VIII. FORMAL SOLUTION OF THE PLASMA PROBLEM

Knowing that we can construct the details of the Green's function let us first examine formally its application to the problem on hand, namely the construction of the functions $_{i}\ldots H_{j}\ldots$ which describe a plasma in a harmonic perturbing field. We take $N = 6$ and interpret

$\underline{x}$ of the general theory as the column vector $\begin{pmatrix} \underline{x} \\ \underline{v} \end{pmatrix}$. Equation (27) is an example of equation (36) with suitable choices of $\underline{a}, \underline{b}$ and c. The solution is therefore as in (40), but we have to provide some additional notation. For the dummy variable $\underline{y}$ of the general theory we now write $\underline{x}'$ for the space part and $\underline{v}'$ for the velocity. For the orbit $\underline{Y}(t)$ we write $\underline{x}' + \underline{X}$ for the space part and $\underline{V}$ for the velocity, so that $\underline{X}, \underline{V}$ is actually the orbit which starts at $\underline{X} = 0$, $\underline{V} = v'$ at time $t = 0$. This is a convenient definition, for $\underline{X}$, $\underline{V}$ now depend only on $(\underline{v}', t)$ and $\underline{x}'$ will soon drop out of the calculation. Specifically, we have

$$\frac{d\underline{X}}{dt} = \underline{V}\ , \quad \frac{d\underline{V}}{dt} = \frac{e}{mc}\, \underline{V} x \underline{B} - \nu \underline{V} \tag{43}$$

In this notation (40) is

$$f(\underline{x},\underline{v},t) = \int\limits_{\underline{x}'} \int\limits_{\underline{v}'} \int\limits_{\tau=0}^{\infty} h(\underline{x}',\underline{v}',t-\tau)\, g[\underline{x}-\underline{x}'-\underline{X}(\underline{v}',\tau), \underline{v}-\underline{V}(\underline{v}',\tau), \tau]\, d^3x'\, d^3v'\, d\tau$$

We assume a dependence like $e^{i(\omega t - \underline{k}\cdot\underline{x})}$ in space and time for f and h, with the usual convention that the problem be solved for $\Im(\omega) > 0$ in the first instance and then analytically continued, the symbols f and h representing the complex amplitudes in the usual way. Also write $\xi = \underline{x}-\underline{x}'$. Then

$$f(\underline{v}) = \iint\limits_{\underline{\xi}\ \underline{v}'} \int\limits_{\tau=0}^{\infty} e^{i(\underline{k}\cdot\underline{\xi}-\omega\tau)} h(\underline{v}')\ g[\underline{\xi}-\underline{X}(\underline{v}',\tau), \underline{v}-\underline{V}(\underline{v}',\tau), \tau]\, d^3\xi\, d^3v'\, d\tau$$

As we want to calculate not $f(\underline{v})$ itself but merely the 0^{th}, 1^{st}, and 2^{nd} moments of it with respect to $\underline{v}$, we construct the Fourier transform

$$I(\underline{\sigma}) = e^{i\underline{\sigma}\cdot\underline{v}} f(\underline{v})d^3v = \int_{\underline{\xi}}\int_{\underline{v}}\int_{\underline{v}'}\int_{\tau=0}^{\infty} e^{i(\underline{k}\cdot\underline{\xi}+\underline{\sigma}\cdot\underline{v}-\omega\tau)} h(\underline{v}')$$

$$\times\, g[\underline{\xi}-\underline{X}(\underline{v}',\tau),\underline{v}-\underline{V}(\underline{v}',\tau),\tau]d^3vd^3v'd^3\xi d\tau$$

The required results are therefore given by I and its derivatives evaluated at $\underline{\sigma}=0$. Holding $\underline{v}'$ and τ fixed, a trivial change of origin in the $\underline{\xi}$ and $\underline{v}$ spaces gives us

$$I(\underline{\sigma}) = \int_{\underline{\xi}}\int_{\underline{v}}\int_{\underline{v}'}\int_{\tau=0}^{\infty} e^{i[\underline{k}\cdot\underline{X}(\underline{v}',\tau)+\underline{\sigma}\cdot\underline{V}(\underline{v}',\tau)-\omega\tau]} h(\underline{v}')$$

$$\times\, e^{i(\underline{k}\cdot\xi+\underline{\sigma}\cdot\underline{v})} g(\underline{\xi},\underline{v},\tau)\; d^3vd^3v'd^3\xi d\tau$$

The integrations with respect to $\underline{\xi}$, $\underline{v}$ can now be carried out, being just the Fourier transform of a gaussian distribution in 6 dimensions; we note that at this point the need to calculate the normalizing function $p(t)$ drops out. The result is

$$I = \int_{\underline{v}'}\int_{\tau=0}^{\infty} e^{i[\underline{k}\cdot\underline{X}(\underline{v}',\tau)+\underline{\sigma}\cdot V(\underline{v}',\tau)-\omega\tau]} h(v')$$

$$\times \exp\left[-\tfrac{1}{2}(\underline{k},\underline{\sigma})\;\underline{r}\left(\frac{\underline{k}}{\underline{\sigma}}\right)\right]d^3v'd\tau$$

where $\underline{r}$ is the 6×6 matrix of the general theory.

Since $\underline{X}$ and $\underline{V}$ actually depend linearly on $\underline{v}'$, let us introduce the abbreviation

$$\underline{k}\cdot\underline{X}(\underline{v}',\tau)+\underline{\sigma}\cdot\underline{V}(\underline{v}',\tau) = \underline{p}\cdot\underline{v}' \qquad (44)$$

so that $\underline{p}(\tau)$ is readily calculated and is linear in $\underline{k}$ and $\underline{\sigma}$. As $h(\underline{v}')$ is to be a maxwellian distribution multiplied by 0.1, or 2 components of $\underline{v}'$, let us calculate I for the case

$$h(\underline{v}') = e^{i\underline{\rho}\cdot\underline{v}'} N_o \left(\frac{m}{2\pi KT_o}\right)^{3/2} e^{-m\underline{v}'^2/2KT_o} \tag{45}$$

so that I and its derivatives with respect to $\underline{\rho}$, all evaluated at $\underline{\rho}=0$, give all the required information. The $\underline{v}'$ integration is now also the Fourier transform of a gaussian function, and carrying it out yields

$$I = N_o \int_{\tau=0}^{\infty} \exp\left\{\frac{-KT_o}{2m}(\underline{p}+\underline{\rho})^2 - \tfrac{1}{2}(\underline{k},\underline{\sigma})\underline{r}\left(\frac{\underline{k}}{\underline{\sigma}}\right) - i\omega\tau\right\} d\tau \tag{46}$$

This is the promised single integral, and is the analogue of the so-called "Gordeyev integral" familiar from earlier work. The exponent is just a quadratic form in $\underline{k},\underline{\rho}$ and $\underline{\sigma}$ whose coefficients are somewhat cumbersome (though elementary) functions of τ. The collision-free theory is of course recovered by setting $\underline{r} = 0$, and there is also some slight simplification in the orbits as $\nu = 0$ in (43). In addition, there is much less to calculate owing to the great simplification of (25) and (26).

We may record here that the H-functions required in (25) and (26) are given explicity in terms of our integral I by

$$_i\ldots H_j\ldots = \left[\frac{1}{i}\frac{\partial}{\partial\sigma_i}\ \ldots\ \frac{1}{i}\frac{\partial}{\partial\rho_j}\ \ldots\ I\right]_{\underline{\sigma}=\underline{\rho}=0} \tag{47}$$

IX. COEFFICIENTS IN THE ABSENCE OF MAGNETIC FIELD

The process will be made clearer by working through the case $\underline{B} = 0$. The unperturbed orbits are given by (cf 43):

$$\frac{d\underline{X}}{dt} = \underline{V}\ , \ \frac{d\underline{V}}{dt} = -\nu\underline{V}$$

so the solution starting at $\underline{x} = 0$, $\underline{V} = \underline{v}'$ is

$$\underline{V} = \underline{v}' e^{-\nu t} \qquad \underline{X} = \frac{1}{\nu} \underline{v}'(1-e^{-\nu t})$$

Comparing with (44),

$$\underline{p} = \frac{1}{\nu}(1-e^{-\nu t})\underline{k} + e^{-\nu t}\underline{\sigma} \tag{48}$$

The matrices $\underline{a}$ and $\underline{b}$ of the general theory are conveniently partitioned into 3 x 3 matrices:

$$\underline{a} = \begin{pmatrix} \underline{0} & \underline{0} \\ \underline{0} & -\eta\underline{I} \end{pmatrix} \qquad \underline{b} = \begin{pmatrix} \underline{0} & \underline{I} \\ \underline{0} & -\nu\underline{I} \end{pmatrix}$$

($\underline{I}$ being the unit matrix).
Thus, the equation for $\underline{r}$, (34), is formally the same as in the one-dimensional example (.7) and we have

$$\underline{r} = \begin{pmatrix} \alpha\underline{I} & \beta\underline{I} \\ \beta\underline{I} & \gamma\underline{I} \end{pmatrix} \tag{49}$$

with α, β, γ as given by (42). So

$$(\underline{k},\underline{\sigma})\ \underline{r}\begin{pmatrix} \underline{k} \\ \underline{\sigma} \end{pmatrix} = \alpha k^2 + 2\beta\underline{k}\cdot\underline{\sigma} + \underline{\sigma}^2 \tag{50}$$

Recalling that $\eta = \nu KT_o/m$ one can combine (46), (48) and (50) to obtain

$$I = N_o \int_0^\infty \exp\left\{-\frac{KT_o}{2m}\left[\frac{2}{\nu^2}(\nu t-1+e^{-\nu t})\underline{k}^2+\underline{\sigma}^2+\underline{\rho}^2+\frac{2}{\nu}(1-e^{-\nu t})(\underline{\sigma}+\underline{\rho})\cdot\underline{k}+2e^{-\nu t}\underline{\sigma}\cdot\underline{\rho}\right] -i\omega t\right\} dt \tag{51}$$

I itself evaluated at $\underline{\sigma}=\underline{\rho}=0$ is therefore

$$N_o \int_0^\infty \exp\left\{-\frac{KTk^2}{m\nu^2}\left(\nu t-1+e^{-\nu t}\right) - i\omega t\right\} dt \tag{52}$$

whilst the derivatives required in (47) involve similar integrals with various other factors preceding the exponential. We may compare (52) with the corresponding integral in the collision-free case,

$$N_o \int_0^N \exp\left\{-\frac{KTk^2t^2}{2m} - i\omega t\right\} dt \tag{53}$$

to which (52) does of course tend as $\nu \to 0$.

(51) is symmetric in $\underline{\rho}$ and $\underline{\sigma}$ so that the H functions are actually the same whether the suffixes go before or after H.

X. COEFFICIENTS IN THE PRESENCE OF A MAGNETIC FIELD

We now consider the full equation (27), and again apply our general technique for obtaining the Green's function. The work is similar to the previous section: we calculate the orbits and solve the equation for $\underline{r}$ by writing $\underline{a}$, $\underline{b}$ and $\underline{r}$ in partitioned form. But the 3 x 3 matrices so introduced are no longer diagonal in ordinary cartesian coordinates. To avoid the excessively tedious calculations which would follow on account of this, we make a transformation to coordinates in which they are diagonal. Namely we define

$$x_1 = (x+iy)/2^{1/2}\ , \quad x^o = z \quad x^{-1} = (x-iy)/2^{1/2} \tag{54}$$

as the new coordinates, denoting the old ones by (x, y, z). Similar formulae hold for components of velocity. The numbering of the new coordinates (due to Buneman, 1961) is particularly convenient as the suffixes can also be used as algebraic quantities. We shall use Greek suffixes such as λ, μ, etc. to label the new coordinates. The transformation (54) is not orthogonal, although it is unitary. We therefore distinguish between contravariant vectors (such as $\underline{x}$ in the above transformation) and covariant vectors, with the usual suffix notation. It is easily verified that the scalar product of two contravariant vectors $\underline{A}$, $\underline{B}$ is $\sum_{\lambda} A^{\lambda} B^{-\lambda}$ in the new coordinates, and this must be identified with $\sum_{\lambda} A^{\lambda} B_{\lambda}$. Clearly the metric tensor is $\delta_{\lambda,-\mu}$ and raising or lowering of the suffixes is achieved by simply changing the sign. (Note that this is not in general the same as taking the complex conjugate, as some of our vector quantities are already complex representatives of harmonic quantities). To convert all the previous results to general tensor notation one must decide for every suffix whether it is covariant or contravariant. This can be done merely by recalling that any pair of contracted suffixes must be one of each type; we shall do this without further comment. A point calling for special care is that the matrix $\underline{b}$ as introduced in (30) is "mixed," having one suffix of each type, and is also non-symmetric, the "first" suffix being the contravariant one. Thus, when the transpose of this is required one should, strictly speaking, use a more cumbersome notation to indicate the order of the suffixes; fortunately we can avoid this as the calculation proceeds immediately to the partitioned matrices, and all the 3 x 3 matrices involved turn out to be symmetric.

In these coordinates the equations for the orbits, (43) become

$$\frac{dX^{\lambda}}{dt} = V^{\lambda} \qquad \frac{dV^{\lambda}}{dt} = -(\nu + i\Omega\lambda)V^{\lambda} \tag{55}$$

where Ω is the gyro-frequency eB/mc.

As this is to be identified with (31) we have at once

$$\underline{b} = \begin{pmatrix} \underline{0} & \underline{I} \\ \underline{0} & -\underline{\Lambda} \end{pmatrix} \tag{56}$$

where $\underline{\Lambda}$ is a diagonal matrix

$$\Lambda^{\lambda}_{\mu} = (\nu+i\Omega\lambda)\delta^{\lambda}_{\mu} \tag{57}$$

The orbit starting at $\underline{X} = 0$, $\underline{V} = \underline{v}$ is therefore

$$X^{\lambda} = \frac{v^{\lambda}}{\nu+i\Omega\lambda}\left[1 - e^{-(\nu+i\Omega\lambda)t}\right] \tag{58}$$

$$V^{\lambda} = v^{\lambda}e^{-(\nu+i\Omega\lambda)t}$$

The defining relation (44) for $\underline{p}$ is

$$p_{\lambda}v^{\lambda} = k_{\lambda}X^{\lambda}+\sigma_{\lambda}V^{\lambda}$$

and so

$$p_{\lambda} = \frac{k_{\lambda}}{\nu+i\Omega\lambda}\left[1 - e^{-(\nu+i\Omega\lambda)t}\right] + \sigma_{\lambda}e^{-(\nu+i\Omega\lambda)t} \tag{59}$$

To find p^{λ} we of course raise the suffix, noting that this involves a change in sign of λ where it appears as an algebraic quantity.

Turning to (34) to be solved for $\underline{r}$, we note that $\underline{r}$ and $\underline{a}$ are both as first introduced, symmetric with both suffixes contra variant. We can write, by analogy with the case of no magnetic field

$$\underline{r} = \begin{pmatrix} \underline{\alpha} & \underline{\beta} \\ \underline{\beta}' & \underline{\chi} \end{pmatrix} \quad \underline{a} = \begin{pmatrix} \underline{0} & \underline{0} \\ \underline{0} & -\underline{\eta} \end{pmatrix} \tag{60}$$

Here $\underline{\beta}$ may be non-symmetric and the form taken by $\underline{\eta}$ in these coordinates will be given shortly. Carrying out the multiplication of partitioned matrices, (34) is

$$\left.\begin{aligned} \frac{d\underline{\alpha}}{dt} - (\underline{\beta}+\underline{\beta}') &= 0 \\ \frac{d\underline{\beta}}{dt} + \underline{\beta}\,\underline{\Lambda} - \underline{\chi} &= 0 \\ \frac{d\underline{\chi}}{dt} - 2\underline{\eta} + \underline{\Lambda}\,\underline{\chi} + \underline{\chi}\,\underline{\Lambda} &= 0 \end{aligned}\right\} \qquad (61)$$

As previously, we can now solve for $\underline{\chi}$, $\underline{\beta}$, $\underline{\alpha}$ in that order, and the fact that we are dealing with matrices presents no difficulty as Λ is diagonal. We note that $\underline{\eta}$, which is $\eta \underline{I}$ in ordinary coordinates, becomes $\eta^{\lambda\mu} = \eta\delta^{\lambda,-\mu}$.

The equation for $\underline{\chi}$ is

$$\frac{d\gamma^{\lambda\mu}}{dt} = 2\eta^{\lambda\mu} + \Lambda^{\lambda}_{\epsilon}\,\gamma^{\epsilon\mu} + \gamma^{\lambda\epsilon}\Lambda^{\mu}_{\epsilon}$$

$$= 2\eta\delta^{\lambda,-\mu} + (\nu+i\Omega\lambda)\gamma^{\lambda\mu} + (\nu+i\Omega\mu)\gamma^{\lambda\mu}$$

i.e.,

$$\frac{d\gamma^{\lambda\mu}}{dt} + [2\nu+i\Omega(\lambda+\mu)]\,\gamma^{\lambda\mu} = 2\eta\delta^{\lambda,-\mu}$$

As $\underline{\chi} = 0$ initially, we can clearly take $\gamma^{\lambda\mu} = 0$ if $\lambda+\mu \neq 0$
If $\lambda+\mu = 0$ we have

$$\frac{d\gamma^{\lambda\mu}}{dt} + 2\nu\gamma^{\lambda\mu} = 2\eta$$

so the complete solution is

$$\gamma^{\lambda\mu} = \frac{\eta}{\nu}\,(1 - e^{-2\nu t})\delta^{\lambda,-\mu} \qquad (62)$$

Apart from the change of coordinates, this result is the same as for no magnetic field (cf. Eq. (42) and (49)). Remembering that $\underline{\chi}$ gives the covariance in the velocity for a set of particles starting at the origin, we conclude simply that the diffusion in velocity space continues to be isotropic in spite of the magnetic field, and this is not surprising as our model is based on an isotropic diffusion term, $-\eta\ \partial^2 f/\partial v_i \partial v_i$.

The solution of (61) for $\beta^{\lambda\mu}$, $\alpha^{\lambda\mu}$ proceeds similarly, and again the components are zero except where $\lambda + \mu = 0$. Here the results differ from those for no magnetic field. We have

$$\frac{d\beta^{\lambda\mu}}{dt} + \beta^{\lambda\epsilon}\Lambda_{\epsilon}^{\ \mu} = \gamma^{\lambda\mu}$$

i.e.,

$$\frac{d\beta^{\lambda\mu}}{dt} + (\nu + i\Omega\mu)\beta^{\lambda\mu} = \frac{\eta}{\nu}\left(1 - e^{-2\nu t}\right)\delta^{\lambda,-\mu}$$

with solution

$$\beta^{\lambda\mu} = \frac{\eta}{\nu}\left\{\frac{1 - e^{-(\nu + i\Omega\mu)t}}{\nu + i\Omega\mu} + \frac{e^{-2\nu t} - e^{-(\nu + i\Omega\mu)t}}{\nu - i\Omega\mu}\right\}\delta^{\lambda,-\mu} \tag{63}$$

Similarly

$$\frac{d\alpha^{\lambda\mu}}{dt} = \beta^{\lambda\mu} + \beta^{\mu\lambda}$$

with solution

$$\alpha^{\lambda\mu} = \frac{2\eta}{\nu^2 + \mu^2\Omega^2}\left\{t + \frac{2e^{-\nu t}(\nu\cos\mu\Omega t - \mu\Omega\sin\mu\Omega t) - 2\nu}{\nu^2 + \mu^2\Omega^2} + \frac{1 - e^{-2\nu t}}{2\nu}\right\}\delta^{\lambda,-\mu} \tag{64}$$

We are now ready to construct our integral I, Eq. (46). The exponent in the integrand contains a quadratic form in the components of $\underline{k}$, $\underline{\rho}$, and σ, which in our new coordinates is

$$\frac{-KT_o}{2m}(p_\lambda+\rho_\lambda)(p^\lambda+\rho^\lambda) - \tfrac{1}{2}(k_\lambda\alpha^{\lambda\mu}k_\mu+2k_\lambda\beta^{\lambda\mu}\sigma_\mu+\sigma_\lambda\gamma^{\lambda\mu}\sigma_\mu)$$

It is convenient to write this as $-\phi -\psi$, where ϕ is the part quadratic in k_λ and ψ is the rest. Then

$$I = N_o \int_0^\infty \exp\left\{-\phi(t) - \psi(t) - i\omega t\right\} dt \tag{65}$$

and assembling the terms with reference to Eq. (59), (62), (63), and (64) yields (noting that $\eta = \nu KT_o/m$)

$$\phi(t) = \frac{KT_o}{m}\left\{\frac{k_\parallel^2}{\nu^2}(\nu t-1+e^{-\nu t}) + \frac{k_\perp^2}{\nu^2+\Omega^2}[\cos x + \nu t-e^{-\nu t}\cos(\Omega t-x)]\right\}$$

$$\psi(t) = \frac{KT_o}{m}\left\{\tfrac{1}{2}\sigma_\lambda\sigma^\lambda+\tfrac{1}{2}\rho_\lambda+\rho^\lambda+k_\lambda\sigma^\lambda\,\frac{1-e^{-(\nu-i\Omega\lambda)t}}{\nu-i\Omega\lambda} + k_\lambda\rho^\lambda\,\frac{1-e^{-(\nu+i\Omega\lambda)t}}{\nu+i\Omega\lambda} + \rho^\lambda\sigma_\lambda e^{-(\nu+i\Omega\lambda)t}\right\} \tag{67}$$

Here $k_\parallel$ and $k_\perp$ are the magnitudes of the components of $\underset{\sim}{k}$ parallel and perpendicular to the magnetic field, so $k_\parallel = k_0$, $k_\perp^2 = 2k_1k_{-1} = k_x^2+k_y^2$. The angle x is such that $\tan \tfrac{1}{2}x = \nu/\Omega$, and $0 \leq x \leq \pi$. Since $\underset{\sim}{\rho}$ and $\underset{\sim}{\sigma}$ are eventually placed equal to zero, ψ always disappears from the exponent in any integration actually carried out, however, we need to know the formula for ψ in order to differentiate I with respect to $\underset{\sim}{\rho}$ or $\underset{\sim}{\sigma}$.

XI. CONSTRUCTION OF THE H-FUNCTIONS

To make use of our fundamental results, Eqs. (24)-(26), we have to construct the various "H" functions required in (25). These in turn are known in terms of I by means of (47). We should first express these results in terms of the $(1,0,-1)$ coordinates with due regard for the position of the suffixes. Since $\underline{u}$ and $\underline{a}$ are naturally contravariant, $\underline{M}$ will be a mixed tensor, with the first suffix contravariant. Hence the correct invariant statements of (24) and (25) are

$$N_o u^\lambda = M^\lambda_\mu \left(a^\mu + \nu u^\mu\right) \tag{68}$$

where

$$M^\lambda_\mu = \frac{m}{KT_o}\left\{ {}^\lambda H_\mu + \frac{3\nu\left({}^\lambda H - \frac{m}{3KT_o}\,{}^\lambda H^\delta_\delta\right)\left(H_\mu - \frac{m}{3KT_o}\,{}^\epsilon H_{\epsilon\mu}\right)}{N_o - 3\nu\left\{H - \frac{m}{3KT_o}\left({}^\epsilon H_\epsilon + H^\delta_\delta\right) + \left(\frac{m}{3KT_o}\right)^2\,{}^\epsilon H^\delta_{\epsilon\delta}\right\}}\right\} \tag{69}$$

and (26) is formally the same. (47) becomes

$$\lambda\ldots H_{\mu\ldots} = \left[\frac{1}{i}\,\frac{\partial}{\partial\,\sigma_\lambda}\,\ldots\frac{1}{i}\,\frac{\partial}{\partial\,\rho^\mu}\,\ldots I\right]_{\underline{\sigma}=\underline{\rho}=0} \tag{70}$$

In other words, the rule is for a suffix before H, apply $\partial/i\partial\sigma$, for a suffix after H, apply $\partial/i\partial\rho$; for a contravariant suffix differentiate with respect to a covariant component and vice versa.

The simplest of these functions is H itself, which is

$$H = N_o \int_0^\infty \exp\left\{-\phi(t) - i\Omega t\right\} dt \tag{71}$$

This is easily seen to reduce to the "Gordeyev" integral in the limit $\nu \to 0$. The additional factors, which have to be inserted for the H's with one or more suffixes, are obtained from derivatives of type (70) with $e^{-\psi}$ playing the part of I. Thus, one has to compute integrals of the type

$$N_o \int_0^\infty F(t) \exp\left\{-\phi(t)-i\omega t\right\} dt \qquad (72)$$

where $F(t)$ has whatever is the appropriate tensor character. We now list the required expressions F:

For ${}^\lambda H_\mu$, F is

$$\frac{KT_o}{m}\, \delta^\lambda_\mu\, e^{-(\nu+i\Omega\lambda)} - \left(\frac{KT_o}{m}\right)^2 k^\lambda k_\mu \frac{\left[1-e^{-(\nu+i\Omega\lambda)t}\right]\left[1-e^{-(\nu+i\Omega\mu)t}\right]}{(\nu+i\Omega\lambda)(\nu+i\Omega\mu)} \qquad (73)$$

For ${}^\lambda H - \frac{m}{3KT_o}\, {}^\lambda H^\delta_\delta$, F is

$$\frac{iKT_o}{3m} k^\lambda \left\{ \frac{KT_o}{m} \frac{1-e^{-(\nu+i\Omega\lambda)t}}{\nu+i\Omega\lambda} \left[\frac{k_\parallel^2}{\nu^2}(1-e^{-\nu t})^2 + \frac{k_\perp^2(1-2e^{-\nu t}\cos\Omega t+e^{-2\nu t})}{\nu^2+\Omega^2} \right] - 2e^{-(\nu+i\Omega\lambda)t} \frac{1-e^{-(\nu-i\Omega\lambda)t}}{\nu-i\Omega\lambda} \right\} \qquad (74)$$

For $H_\mu - \frac{m}{3KT_o}\, {}^\lambda_\lambda H_\mu$, F is the same as (74) with λ replaced by μ and k^λ by k_μ (but no changes of sign in the suffixes).

For $H - \frac{m}{3KT_o}\left({}^\epsilon_\epsilon H + H^\delta_\delta\right) + \left(\frac{m}{3KT_o}\right)^2 {}^\epsilon_\epsilon H^\delta_\delta$, F is

$$\frac{2}{3} e^{-2\nu t} - \frac{4KT_o}{9m} e^{-\nu t} \left\{ \frac{k^2(1-e^{-\nu t})^2}{\nu^2} + \frac{k_\perp^2}{\nu^2+\Omega^2}\left[\frac{\nu^2-\Omega^2}{\nu^2+\Omega^2}\left((1+e^{-2\nu t})\cos\Omega t-2e^{\nu t}\right) + \frac{2\nu\Omega}{\nu^2+\Omega^2}(1-e^{-\nu t}\sin\Omega t)\right]\right\} + \left(\frac{KT_o}{3m}\right)^2 \left\{ \frac{k_\parallel^2}{\nu^2}(1-e^{-\nu t})^2 + \frac{k_\perp^2}{\nu^2+\Omega^2}\left(1-2e^{-\nu t}\cos\Omega t+e^{-2\nu t}\right)\right\}^2 \qquad (75)$$

Though these expressions are somewhat cumbersome, calculation of all the integrals in Eq. (69) by computer seems quite feasible.

To recover the results for no magnetic field one simply sets $\Omega = 0$ and regards λ and μ as ordinary cartesian suffixes; the position of the suffixes being now irrelevant. Placing $\underline{k}$ along a coordinate axis makes M_{ij} diagonal, thus separating longitudinal and transverse effects, so that no elaborate matrix calculations are necessary. This will not, however, be pursued here, as our main interest lies in asking how easily the collisions destroy gyro-resonance effects.

XII. PROPERTIES OF THE INTEGRALS

For the quantitive solution of any problem such as the dispersion equation for plasma oscillations or instabilities, or the calculation of spectra in incoherent scatter, there seems no alternative to numerical work; however, some crude estimates of what can be expected in various situations can easily be made. To do this we consider the integral for H, as given by (71), and its dependence on ω, $\underline{k}$, Ω and ν. The more elaborate integrals (72) can be expected to behave in much the same way.

As an example, let us consider wave vectors perpendicular to $\underline{k}$, so that $k_{\parallel} = 0$. Then from (66)

$$\phi(t) = \frac{KTk^2}{m(\Omega^2+\nu^2)}\left[\cos x + \nu t - e^{-\nu t}\cos(\Omega t - x)\right] \qquad (76)$$

Now when $\nu = 0$ so that $x = 0$, this becomes

$$\phi(t) = \frac{KTk^2}{m\Omega^2}(1 - \cos\Omega t) \qquad (77)$$

This is periodic, so the integral

$$\int_0^\infty \exp \quad - \phi(t) - i\omega t \quad dt \tag{78}$$

diverges for $\omega = n\Omega$ $(n = 0,1,2\ldots)$, and this is the origin of the gyroresonance effect. On reintroducing a small but non zero value of $k_{\|}$ the divergence is removed and a rough way to estimate the value of $k_{\|}$ (or, equivalently, the direction of $\underline{k}$) at which the resonance is practically removed is given by Farley, Dougherty and Barron, 1961 (see especially p. 253); this estimate agrees well with actual computations. A similar procedure can be applied to (76): this time we keep $k_{\|} = 0$ but increase ν from zero. Again the additional factors so introduced limit, and eventually remove, the gyro-resonances.

Suppose first that $0 < \nu \ll \Omega$. Then on carrying out an integration such as (78) the factor $e^{-\nu t}$ entering (76) is relatively unimportant, for it permits many oscillations of the cosine term before appreciably reducing its amplitude. The term νt in (78) may however, be important, for it introduces a factor

$$\exp \quad -\left(\frac{KTk^2 \nu t}{m\Omega^2}\right)$$

in the integrand of (78). The question is whether this factor permits many oscillations of the integrand before reducing it severly. The condition for this is clearly $KTk^2\nu/m\Omega^3 \ll 1$, i.e.,

$$\nu \ll \Omega^3 m/KTk^2 \tag{79}$$

If $KTk^2/m\Omega^2$ is not much greater than unity, (79) is already satisfied by the hypothesis that $\nu \ll \Omega$. In that case the gyro-resonance effect can only be destroyed by increasing ν until $\nu \sim \Omega$. On the other hand, if $KTk^2/m\Omega^2 \gg 1$, (79) may be violated even when $\nu \ll \Omega$. The full condition for the survival of an appreciable gyro-resonances effect is thus

$$\nu \ll \Omega \quad \text{or} \quad \Omega^3 m/KTk^2, \quad \text{whichever is the less} \tag{80}$$

This contrasts sharply with the criterion arising for collisions with neutral molecules. Using the BGK model, the integrals entering the calculations are all of the Gordeyev type but with ω replaced by $\omega - i\nu$. This leads to the sole condition $\nu \ll \Omega$, (cf. Dougherty and Farley, 1963).

XIII. IONOSPHERIC APPLICATIONS

To discuss the scattering of radar waves in the ionosphere one needs to know the behavior of a plasma at a value of $\underline{k}$ fixed by the experiment, and for a wide range of real values of ω. In the case of incoherent scattering the perturbations are simply the fluctuations arising in thermal equilibrium, and the collision-free theory has been worked out in great detail. For the other much stronger forms of scattering, the source of the perturbations is unknown but it seems reasonable to expect that any explanation of its spectral behavior will again be a matter for Boltzmann's equation. The question arises how to include collisions in such calculations. For the collisions with neutral particles the BGK model seems adequate, and has been exploited by Dougherty and Farley (1963). For heights up to about 120 km, the ion-neutral collisions have an appreciable effect because $\nu_{in} > \Omega_i$. Gyro-resonance effects would not therefore be expected. In the F region, ν_{in} is quite negligible, yet attempts to observe the gyro-resonance in incoherent scatter at the equator have failed.

To account for this Farley (private communication) made the suggestion that it is ion-ion collisions which eradicate the gyro-resonance effect in the F-region, notwithstanding that the relevant collision frequency, ν_{ii} is smaller than Ω_i. The reason for this is of course that it is the second alternative in (80) which applies and is not satisfied. Typical figures for the ionosphere might be as follows. Taking $T = 1200^{\circ}$, the atomic weight as 16, $\log \Lambda \simeq 13$ and $N_1 = 10^6$, (13) gives $\nu_{ii} \simeq 7\ \text{sec}^{-1}$, while $\Omega_i \sim 160\ \text{sec}^{-1}$. For experiments at 50 Mc/sec, $K(=4\pi/\text{wave-length})$ is 2.10^{-2} cm^{-1}, while $(KT/m)^{1/2}$ is about 8.10^4 cm/sec. Hence $\Omega^3 m/KTk^2 \sim 1.6$ sec^{-1}. Here, then, ν lies between the two quantities mentioned in (80). Actually the interesting case in incoherent scatter occurs when $\Omega_i < k(KT/m_i)^{1/2}$. For if $\frac{k}{\Omega_i}\left(\frac{KT}{m_i}\right)^{1/2} = n$, then n is roughly the number of harmonics of the ion gyro-frequency which would have a pronounced resonance peak in the spectrum; in the example just quoted $n = 10$. But the second quantity in (80) is Ω/n^2, which then necessarily gives the stronger inequality.

Another type of scattering is the aspect sensitive scattering observed at Stanford. Recently Colin, Burns and Eshleman (1963) have reported the detection of a weak but clearly identifiable component of the returned signal with Doppler shift equal to the gyro-frequency of NO^+ ions. Higher harmonics of the gyro-frequency are not observed. So far this effect appears to be restricted to the night-time E region. Here an atomic weight of 30 is appropriate, and temperature about 300^o, and the operating frequency is 23 Mc/sec. Neverthless the figures are not very different from those just quoted, if N_i is again 10^6. But in fact N_i decreases at night in the E region, and according to (13) this decreases ν_{ii}, perhaps to an extent that (30) is now satisfied, though by a not very strong inequality. In this case one would expect a weak gyro-resonance effect to appear, as indeed it does.

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